Transformada cuántica de Fourier

Computación Cuántica: Algorítmica y Software

Joaquín Keller

2026 UCV

Transformada cuántica de Fourier

$$\hat{\mathcal{F}}\left|\underline{k}\right\rangle = \frac{1}{\sqrt{2^n}}\sum_{j=0}^{2^n-1} e^{2i\pi\frac{kj}{2^n}}\left|\underline{j}\right\rangle \qquad 0 \le k < 2^n$$

Factorización: $$\boxed{\hat{\mathcal{F}}\left|\underline{k}\right\rangle = \frac{1}{\sqrt{2^n}}\bigotimes_{\ell=1}^{n}\Big(|0\rangle + e^{2i\pi\frac{k}{2^\ell}}|1\rangle\Big)}$$

$$= \frac{1}{\sqrt{2^n}}\Big(|0\rangle + e^{2i\pi\frac{k}{2}}|1\rangle\Big)\otimes\Big(|0\rangle + e^{2i\pi\frac{k}{2^2}}|1\rangle\Big)\otimes \ldots \otimes\Big(|0\rangle + e^{2i\pi\frac{k}{2^n}}|1\rangle\Big)$$

$$= \frac{1}{\sqrt{2^n}}\sum_{j=0}^{2^n-1}\gamma_j\,|j\rangle \qquad \left|\underline{j}\right\rangle = \bigotimes_{\ell=1}^{n}\left|j_{n-\ell}\right\rangle = \left|j_{n-1}j_{n-2}\ldots j_2 j_1 j_0\right\rangle$$

$$= \frac{1}{\sqrt{2^n}}\sum_{j=0}^{2^n-1}\bigotimes_{\ell=1}^{n} e^{2i\pi\frac{k}{2^\ell}j_{n-\ell}}\left|j_{n-\ell}\right\rangle \qquad e^{\alpha j_i}|j_i\rangle = \begin{cases}|0\rangle & \textbf{si } j_i = 0\\ e^{\alpha}|1\rangle & \textbf{si } j_i = 1\end{cases}$$

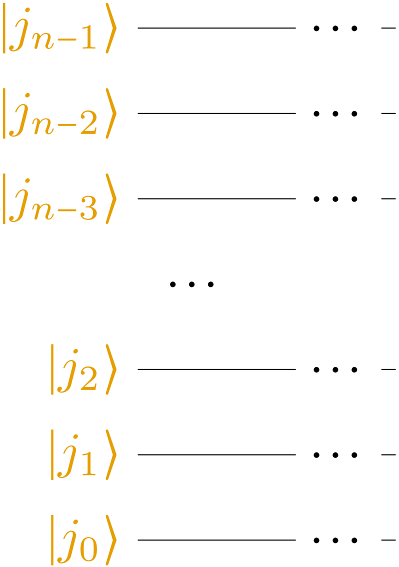
$$\left|\underline{j}\right\rangle = \bigotimes_{\ell=1}^{n} \left|j_{n-\ell}\right\rangle \qquad j = \sum_{\ell=0}^{n-1} j_\ell 2^\ell$$

$$\hat{\mathcal{F}}\left|\underline{k}\right\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \bigotimes_{\ell=1}^{n} e^{2i\pi\frac{kj_{n-\ell}}{2^\ell}} \left|j_{n-\ell}\right\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} \left(\prod_{\ell=1}^{n} e^{2i\pi\frac{kj_{n-\ell}}{2^\ell}}\right) \bigotimes_{\ell=1}^{n} \left|j_{n-\ell}\right\rangle$$

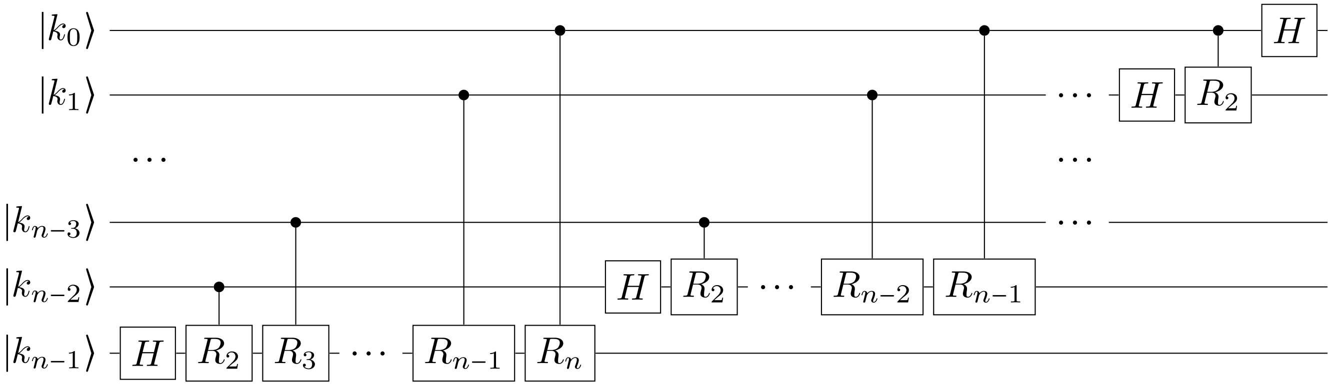
$$\prod_{\ell=1}^{n} e^{2i\pi\frac{kj_{n-\ell}}{2^\ell}} = e^{\left(\sum_{\ell=1}^{n} 2i\pi\frac{kj_{n-\ell}}{2^\ell}\right)}$$

$$\sum_{\ell=1}^{n} 2i\pi\frac{kj_{n-\ell}}{2^\ell} = \frac{2i\pi k}{2^n} \sum_{\ell=1}^{n} \frac{2^n j_{n-\ell}}{2^\ell} = \frac{2i\pi k}{2^n} \sum_{\ell=1}^{n} j_{n-\ell} 2^{n-\ell} = \frac{2i\pi k}{2^n} \sum_{\ell=0}^{n-1} j_\ell 2^\ell = \frac{2i\pi k}{2^n} j$$

$$\hat{\mathcal{F}}\left|\underline{k}\right\rangle = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} e^{\frac{2i\pi k}{2^n} j} \left|\underline{j}\right\rangle$$


$|j_{n-1}\rangle$
$|j_{n-2}\rangle$
$|j_{n-3}\rangle$
$\cdots$
$|j_2\rangle$
$|j_1\rangle$
$|j_0\rangle$

Circuito para QFT



$$R_\ell = \begin{pmatrix} 1 & 0 \\ 0 & 2^{\frac{2i\pi}{2^\ell}} \end{pmatrix}$$

QFT: $\mathcal{O}(n^2)$ vs FFT: $\mathcal{O}(n2^n)$=$\mathcal{O}(N\log N)$

[nota: se necesita un swap de todos los qubits]

Demostración

$$\hat{\mathcal{F}}\,|\underline{k}\rangle = \frac{1}{\sqrt{2^n}} \bigotimes_{\ell=1}^{n} \Big(|0\rangle + e^{2i\pi\frac{k}{2^\ell}}\,|1\rangle\Big) \qquad\qquad |\underline{k}\rangle = \bigotimes_{\ell=1}^{n} |k_{n-\ell}\rangle$$

$$H^{n-1}\,|k\rangle = \Big(\bigotimes_{\ell=1}^{n-1} |k_{n-\ell}\rangle\Big) \otimes \frac{1}{\sqrt{2}}\big(|0\rangle + e^{\frac{2i\pi}{2}}\,|1\rangle\big)$$

et cætara...

En otra sesión se estudiará el algoritmo de estimación de fase cuántica