Compuertas de un cubit

Computación Cuántica: Algorítmica y Software

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Las compuertas de Pauli

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_0 = I \qquad \sigma_x = \sigma_1 = X \qquad \sigma_y = \sigma_2 = Y \qquad \sigma_z = \sigma_3 = Z$$

Las compuertas de Pauli forman un grupo

$$XX = YY = ZZ = I$$

$$XY = -YX = iZ \qquad \Longrightarrow \quad XY \backsimeq YX \backsimeq Z$$

$$YZ = -ZY = iX \qquad \Longrightarrow \quad YZ \backsimeq ZY \backsimeq X$$

$$ZX = -XZ = iY \qquad \Longrightarrow \quad ZX \backsimeq XZ \backsimeq Y$$

$$XYZ = iI \qquad \Longrightarrow \quad XYZ \backsimeq I$$

Forma polar exponencial

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$(e^{i\theta})^2 = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta$$

formulas ángulo doble: $\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin 2\theta = 2\cos\theta\sin\theta$

$$= \cos 2\theta + i\sin 2\theta = e^{i\,2\theta}$$

Forma polar exponencial para matrices

$$A^2 = I \qquad \boxed{e^{i\theta A} = \cos\theta \; I + i\sin\theta \; A}$$

$$(e^{i\theta A})^2 = (\cos\theta \; I + i\sin\theta \; A)^2 = \cos^2\theta I + 2i\cos\theta\sin\theta A + i^2\sin^2\theta A^2$$

$$= \cos^2\theta I + 2i\cos\theta\sin\theta A - \sin^2\theta I = e^{i\,2\theta A}$$

$$X^2 = Y^2 = Z^2 = I$$

Rotaciones de un cubit

$$R_x(\theta) = X_\theta = e^{-i\frac{\theta}{2}X}$$

$$R_y(\theta) = Y_\theta = e^{-i\frac{\theta}{2}Y}$$

$$R_z(\theta) = X_\theta = e^{-i\frac{\theta}{2}Z}$$

Rotación X

$$R_x(\theta) = e^{-i\frac{\theta}{2}X} = \cos(-\tfrac{\theta}{2})I + i\sin(-\tfrac{\theta}{2})X \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \cos\tfrac{\theta}{2}I - i\sin\tfrac{\theta}{2}X = \cos\tfrac{\theta}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\sin\tfrac{\theta}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\boxed{R_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}}$$

$$X_{2\pi} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \simeq I$$

$$X_{\pi} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -iX \simeq X$$

Rotación Y

$$\boxed{R_y(\theta) = e^{-i\frac{\theta}{2}Y} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}}$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Y_{2\pi} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \backsimeq I$$

$$Y_{\pi} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -iY \backsimeq Y$$

Rotación Z

$$\boxed{R_z(\theta) = e^{-i\frac{\theta}{2}Z} = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$R_z(\theta) \backsimeq e^{i\frac{\theta}{2}} \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

[“Phase shift”]

Potencias de X, Y, Z

$$X^a = Rx(a\pi) = \left(e^{i\pi X}\right)^a = e^{ia\pi X}$$
$$Y^a = Ry(a\pi)$$
$$Z^a = Rz(a\pi)$$

Ejemplo: $\sqrt{NOT} = X^{1/2}$

Mathematica™: `MatrixPower`

Compuerta de Hadamard

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad H^2 = I$$

$$H X_\theta H = Z_\theta \qquad H Y_\theta H = Y_{-\theta} \qquad H Z_\theta H = X_\theta$$

$$H \backsimeq Y^{1/2} Z$$

$$H\left|0\right\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$ [“Superposición”]

Las variantes de la compuerta de Hadamard

$$
\begin{array}{llll}
H_1 = Z^{1/2}\ H Z^{-1/2} & H_1 Y_\theta H_1 = Z_\theta & H_1 X_\theta H_1 = X_{-\theta} & H_1 Z_\theta H_1 = Y_\theta \\
H_2 = Z^{-1/2} H Z^{1/2} & \ldots & & \\
H_3 = X^{1/2}\ H X^{-1/2} & \ldots & & \\
H_4 = X^{-1/2} H X^{1/2} & \ldots & & \\
H_5 = Y^{1/2}\ H Y^{-1/2} & \ldots & & \\
H_6 = Y^{-1/2} H Y^{1/2} & \ldots & &
\end{array}
$$

Ejercicio: completar

Una de esas es la compuerta de Hadamard?

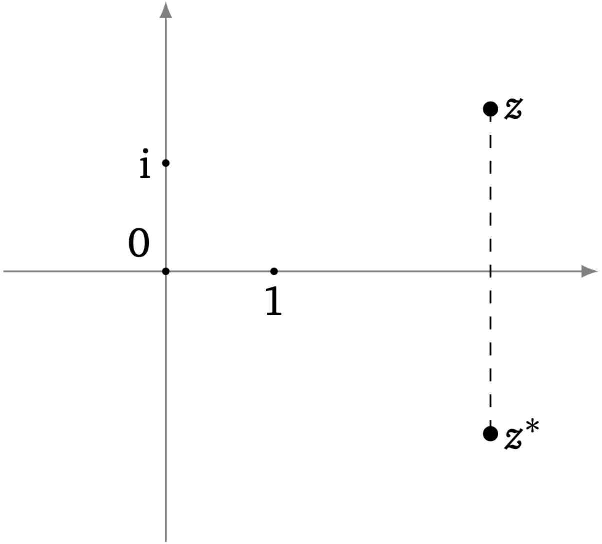
Conjugado

$$z \in \mathbb{C} \qquad z = a + ib \qquad a, b \in \mathbb{R}$$

$$\overline{z} = z^* = \overline{a + ib} = (a+ib)^* = a-ib$$

$$zz^* = (a+ib)(a-ib) = a^2-(ib)^2 = a^2+b^2$$

$$z^{**} = z \qquad zz^* = |z|^2$$



Notación bra

$$|\psi\rangle = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$ vector columna o matriz $n \times 1$

$$\langle\psi| = (x_1^* \ x_2^* \ . \ . \ x_n^*)$$ covector o matriz $1 \times n$

Notación equivalente: $|\psi\rangle^* = \langle\psi|$

$$\langle\psi|\psi\rangle = \sum_{i=1}^{n} x_i x_i^* = \|\psi\|^2$$

Transposición o traspuesta de una matriz

$$A^T = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

Las columnas son las líneas

Mathematica™: `Transpose`

Matriz traspuesta conjugada

$$A^\dagger = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}^\dagger = \begin{pmatrix} a^* & d^* \\ b^* & e^* \\ c^* & f^* \end{pmatrix}$$

Notación equivalente (no utilizada en cuántica): A^*

LaTeX: `\dagger` $->$ $\dagger$ Mathematica™: `ConjugateTranspose`

Algunas propiedades y notaciones

$$\langle u|\lambda v\rangle = \langle \lambda^* u|v\rangle = \lambda \langle u|v\rangle \qquad \langle u|v\rangle = \langle v|u\rangle^*$$

$$(AB)^\dagger = B^\dagger A^\dagger$$

$$\langle u|Av\rangle = \langle A^\dagger u|v\rangle \qquad [\ u^* A v = (u^* A) v = (A^* u)^* v\ ]$$

$$\langle u|A|v\rangle = \langle u|Av\rangle = \langle A^\dagger u|v\rangle$$

$|\psi\rangle^\dagger = \langle\psi|$ [esta notación no se utiliza]

Matrices unitarias

Definición: $UU^\dagger = I$ es decir $U^\dagger = U^{-1}$

$$UU^\dagger = I \implies \langle Uv|Uv\rangle = \langle U^\dagger Uv|v\rangle = \langle v|v\rangle$$

$$\implies \|Uv\| = \|v\|$$

$\|Uv\| = \|v\| \implies UU^\dagger = I$ [demostración como ejercicio]

las compuertas cuánticas son unitarias